INT. J. NUM. METH. HEAT FLUID FLOW, VOL. 2, 391–406 (1992)

SOLUTION OF INVERSE STEFAN PROBLEMS BY A SOURCE-AND-SINK METHOD

C. K. HSIEH, MEHDI AKBARI AND HONGJUN LI

Department of Mechanical Engineering, University of Florida, Gainesville, FL 32611, USA

ABSTRACT

A method has been developed for the solution of inverse heat diffusion problems to find the initial condition, boundary condition, and the source and sink function in the heat diffusion equation. The method has been used in the development of a source-and-sink method to find the boundary conditions in inverse Stefan problems. Green's functions have been used in the solution, and the problems are solved by using two approaches: a series solution approach, and a time incremental approach. Both can be used to find the boundary conditions without reliance on the flux information to be supplied at both sides of the interface. The methods are efficient in that they require less equations to be solved for the conditions. The numerical results have shown to be accurate, convergent, and stable. Most of all, the results do not degrade with time as in other time marching schemes reported in the literature. Algorithms can also be easily developed for the solution of the conditions.

KEY WORDS Inverse Stefan problems Heat diffusion Green's functions

INTRODUCTION

Inverse solution techniques have received much attention in recent years. For example, in the solution of heat diffusion problems, the regular problems have been those that solve the temperature field in a system domain in which the heat transfer is governed by a specified diffusion process and subjected to given initial and boundary conditions. As for the inverse problems, the roles of known and unknown quantities are exchanged. Extra temperatures are usually given either at the boundary or at interior points, and these temperatures are used together with the other conditions to solve for the unknowns, including the initial condition[1], boundary conditions[2], boundary positions[3], heat source or sink functions[4], and even the properties in the heat diffusion equation[5]. These problems are ill posed, and the solution techniques developed for them have found to be highly useful in that they are even applicable to the solution of certain types of regular problems which are difficult to solve in their traditional forward approach; see Reference 6 for example.

There have been numerous efforts developed in the literature for the solution of the regular problems. Solution of the inverse problems, however, is a more recent endeavour; not many methods have been developed and those in use may not be sufficiently effective in the sense that their accuracy may not be high or their solution may hinge on the supply of extra conditions for input. It is thus the purpose of this paper to develop a solution technique that is effective and easy to implement numerically. It will be applied to the solution of one- and two-phase Stefan problems in which the interface motion is specified, and the problems are solved for the boundary conditions that provide for this interface motion. Such problems are found in a variety

0961–5539/92/050391–16$2.00

Received October 1991
Revised June 1992

of engineering applications. For example, in metallurgy or heat treatment process, the mechanical properties of a cast or heat treatment body are determined by the rate of motion of the solidification interface[7]. The inverse problems are also important in the enhancement of the strength of fusion welding[8], control of the surface coating in fluidized beds[9], and prediction of the biological tissue destruction and preservation in freezing–thawing cycles[10], among others.

MOTIVATION

Heat diffusion in a medium with constant properties is governed by the partial differential equation:

$$\nabla^2 T(\bar{r},t) + \frac{u'''(\bar{r},t)}{k} = \frac{1}{\alpha}\frac{\partial T(\bar{r},t)}{\partial t} \qquad \begin{matrix} \bar{r} \in R \\ t > 0 \end{matrix} \tag{1}$$

where all notations have their usual meaning. For this medium, the conditions imposed on the boundary are usually one of the following types:

$$T(\bar{r}_i,t) = F_i(\bar{r}_i,t) \qquad \bar{r}_i \in B_i \tag{2}$$

$$\frac{\partial T(\bar{r}_i,t)}{\partial n_i} = -\frac{G_i(\bar{r}_i,t)}{k_i} \qquad \bar{r}_i \in B_i \tag{3}$$

$$\frac{\partial T(\bar{r}_i,t)}{\partial n_i} + \frac{h_i}{k_i}T(\bar{r}_i,t) = \frac{1}{k_i}H_i(\bar{r}_i,t) \qquad \bar{r}_i \in B_i \tag{4}$$

which represent the familiar Dirichlet, Neumann, and Robin conditions, respectively. In (3) and (4), n_i denotes an outward drawn normal. Then, with the additional initial condition given as:

$$T(\bar{r},0) = T_i(\bar{r}) \tag{5}$$

the temperature can be solved by means of Green's function as[11]:

$$T(\bar{r},t) = \int_{R'} G(\bar{r},t\,|\,\bar{r}',0)T_i(\bar{r}')\,\mathrm{d}V' + \frac{\alpha}{k}\int_0^t \int_{R'} G(\bar{r},t\,|\,\bar{r}',\tau)u'''(\bar{r}',\tau)\,\mathrm{d}V'\,\mathrm{d}\tau + \sum_i \left\{ \quad \right\} \tag{6}$$

Here, the braced term is used to account for the three boundary conditions given earlier. Their expressions are listed in *Table 1*.

A distinct feature is found in the Green's function method above—the effects of the initial condition, heat generation (or destruction), and boundary conditions are embodied respectively in the first, second, and third terms on the right of (6). Then, in the case of regular problems, once these conditions are fully specified, the temperature can be easily found. In this effort, the Green's function can be obtained by using the concept of point charges[11] or by solving auxiliary problems[12].

Table 1 Expressions to account for effects of boundary conditions

Boundary condition	{ } Expression
$T(\bar{r}_i',t) = F_i(\bar{r}_i',t)$	$-\alpha \int_0^t \int_{S_i'} \frac{\partial G(\bar{r},t\,|\,\bar{r}_i',\tau)}{\partial n_i'} F_i(\bar{r}_i',\tau)\,\mathrm{d}S_i'\,\mathrm{d}\tau$
$\frac{\partial T(\bar{r}_i',t)}{\partial n_i} = -\frac{G_i(\bar{r}_i',t)}{k_i}$	$-\alpha \int_0^t \int_{S_i'} G(\bar{r},t\,|\,\bar{r}_i',\tau)\frac{G_i(\bar{r}_i',\tau)}{k_i}\,\mathrm{d}S_i'\,\mathrm{d}\tau$
$\frac{\partial T(\bar{r}_i',t)}{\partial n_i} + \frac{h_i}{k_i}T(\bar{r}_i',t) = \frac{1}{k_i}H_i(\bar{r}_i',t)$	$\alpha \int_0^t \int_{S_i'} G(\bar{r},t\,|\,\bar{r}_i',\tau)\frac{H_i(\bar{r}_i',\tau)}{k_i}\,\mathrm{d}S_i'\,\mathrm{d}\tau$

As it turns out, the format of (6) is particularly suited for the solution of the inverse problems. For the inverse problems, the left hand side of this equation can be used to represent the extra temperature that is provided either at the boundary or at an interior point. This temperature can then be used to determine the missing information, such as the initial condition, heat generation, or boundary conditions. In these efforts, the missing quantities can be expanded either in a polynomial or an infinite series, which is substituted into their respective integrals in (6). The resulting equation is then solved numerically for the coefficients in the series to complete the solution. This method clearly works for the initial condition and the heat generation. However, for the boundary condition, since it is unknown *a priori*, one must first assume a particular 'type' of condition that is imposed on the boundary. For convenience, one could use (2) or (3) for this condition. Green's function is then found with this assumed condition. Notice that the boundary condition can be exchanged if necessary as shown in Reference 13. That is to say, a Dirichlet condition can be accomplished by means of a Neumann condition and *vice versa*. The search for the condition is thus not restricted by the type of the conditions assumed. Better yet, an incremental solution approach can be developed to track the boundary conditions accurately as will be shown later. The concept above will now be applied for the solution of inverse Stefan problems in which the interface positions are given, and these positions are used to find the boundary conditions that must be imposed to cause the interface motions.

GENERAL ANALYSIS

For the sake of illustration in what follows, the Stefan problems consist of two stages: a pre-melt stage, when heat is added to the surface of a subcooled medium to raise its temperature to the phase-change temperature; and a melting stage, when the medium changes phase and the melting starts at the surface and the interface moves inward with time. It is assumed that the properties for different phases are constant and of equal values. The medium has a distinct melting temperature; that is, no mushy zone in the medium. Convection is negligible. For generality, the analysis will be developed for the solution of both melting and solidification problems. The analysis can also be extended for the solution of Stefan problems in multiple phases and in a medium with unequal phase properties as will be discussed later. For the moment, the simplified problems will be solved by considering the medium shown in *Figure 1*. The formulation of these problems follows below.

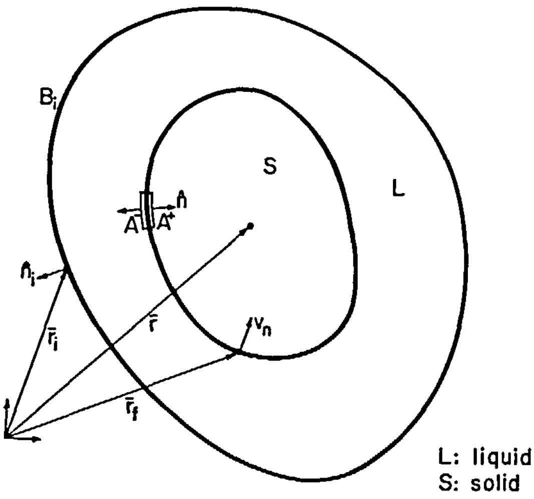

Figure 1 System analysed

Pre-melt stage

Governing equation:

$$\nabla^2 T_0(\bar{r}, t) = \frac{1}{\alpha}\frac{\partial T_0(\bar{r}, t)}{\partial t} \qquad \bar{r} \in (L \cup S), \quad t_0 \geqslant t > 0 \tag{7}$$

Initial condition:

$$T_0(\bar{r}, 0) = T_i(\bar{r}) \tag{8}$$

Melting stage

Liquid region:
Governing equation:

$$\nabla^2 T_L(\bar{r}, t) = \frac{1}{\alpha}\frac{\partial T_L(\bar{r}, t)}{\partial t} \qquad \bar{r} \in L, \quad t > t_0 \tag{9}$$

Solid region:
Governing equation:

$$\nabla^2 T_S(\bar{r}, t) = \frac{1}{\alpha}\frac{\partial T_S(\bar{r}, t)}{\partial t} \qquad \bar{r} \in S, \quad t > t_0 \tag{10}$$

Initial condition:

$$T_S(\bar{r}, t_0) = T_0(\bar{r}, t_0) \tag{11}$$

Interface conditions:

$$T_L(\bar{r}_f, t) = T_m = T_S(\bar{r}_f, t) \tag{12}$$

$$\frac{\partial T_S(\bar{r}_f, t)}{\partial n} - \frac{\partial T_L(\bar{r}_f, t)}{\partial n} = \frac{\rho L}{k} v_n(t) \tag{13}$$

$$v_n(t) = \bar{V} \cdot \hat{n} \tag{14}$$

Here $\bar{r}_f$ denotes the interface position and v_n the history of the interface motion. For the inverse Stefan problems of interest in this study, this history is used for the determination of the missing boundary conditions.

The problems as posed can be solved by use of the Green's function method described in the preceding section. However, a direct use of this method would require (6) to be applied to two separate regions, liquid and solid, and the solution so obtained may not be as efficient as one desires. A source and sink method is thus used[14–16]. In this method, the melting interface is taken to be a moving heat-sink front and a freezing interface is taken to be a moving heat-source front. Then, in sharp contrast to the conventional methods in which different equations are used to represent the temperatures in different regions, only one equation will be derived. Whether it is in the solid or liquid region is determined by the position that is assigned in the temperature equation. The solution of the inverse problems can then be simplified with this method. Following this approach, the melting stage is solved by considering an equivalent problem as follows.

Governing equation for the equivalent problem:

$$\nabla^2 T(\bar{r}, t) \pm \frac{\rho L}{k} v_n(t)\delta(\bar{r} - \bar{r}_f) = \frac{1}{\alpha}\frac{\partial T(\bar{r}, t)}{\partial t} \qquad \bar{r} \in (L \cup S), \quad t > t_0 \tag{15}$$

Initial condition for the equivalent problem:

$$T(\bar{r}, t_0) = T_0(\bar{r}, t_0) \tag{16}$$

Interface conditions for the equivalent problem:

$$T(\bar{r}_f, t) = T_m, \qquad v_n(t) = \bar{V} \cdot \hat{n} \tag{17a, b}$$

where $\delta(\bar{r} - \bar{r}_f)$ denotes a Dirac delta function. The signs preceding this function are used for freezing (+) and melting (−).

It can be shown readily that (15) reduces to (9) and (10). Furthermore, by integrating (15) across the interface from $\bar{r}_f - e$ to $\bar{r}_f + e$ and forcing e to be zero in a limiting process, (15) reduces to (13). This can be proved by using the pill box at the interface in *Figure 1*. Other equivalences for the melting stage are apparent.

The equivalent problem can be solved by referring to (6) in which the heat generation term is changed to the interface motion term as:

$$T(\bar{r}, t) = \int_{L \cup S} G(\bar{r}, t \,|\, \bar{r}', 0) T_i(\bar{r}')\, \mathrm{d}V' \pm \frac{L}{c} \int_{t_0}^{t} \int_{L \cup S} G(\bar{r}, t \,|\, \bar{r}', \tau) v_n(\tau) \delta(\bar{r}' - \bar{r}_f)\, \mathrm{d}V'\, \mathrm{d}\tau + \sum_i \Big\{ \quad \Big\} \tag{18}$$

where the plus sign is used for freezing and minus sign for melting. Finally, the boundary conditions can be found by setting $\bar{r}$ in this equation to the interface position, $\bar{r}_f$, and $T(\bar{r}_f, t)$ to the melting temperature, T_m, as:

$$T_m = \int_{L \cup S} G(\bar{r}_f, t \,|\, \bar{r}', 0) T_i(\bar{r}')\, \mathrm{d}V' \pm \frac{L}{c} \int_{t_0}^{t} \int_{L \cup S} G(\bar{r}_f, t \,|\, \bar{r}', \tau) v_n(\tau) \delta(\bar{r}' - \bar{r}_f)\, \mathrm{d}V'\, \mathrm{d}\tau + \sum_i \Big\{ \quad \Big\} \tag{19}$$

The missing boundary conditions can then be found by solving them implicitly. In this effort, the time when melting starts (t_0) can be determined by solving the pre-melt problem, whose solution can again be taken to be the special case of (6) in which the heat generation term is zero. Solution in this stage is thus elementary.

EXAMPLES

The analysis above is now used to solve example problems which are in semi-infinite domain in which the interface motion is given. For the sake of generality, the analysis will be developed for determining either the Dirichlet condition or the Neumann condition that is imposed on the boundary at $x = 0$, and the interface motion may be the result of either melting or solidification in a one- and two-phase medium. Also for illustration purposes, the heat flow is one-dimensional, the initial temperature being uniform. Extensions to more dimensions are provided in the Appendix.

For the problem given, Green's function can be found to be:

$$G(x, t \,|\, x', \tau) = \frac{1}{2\sqrt{\pi\alpha(t - \tau)}} \left[\exp\left(-\frac{(x - x')^2}{4\alpha(t - \tau)} \right) \pm \exp\left(-\frac{(x + x')^2}{4\alpha(t - \tau)} \right) \right] \tag{20}$$

where the plus and minus signs are to be used when the flux and temperature condition is assumed to appear at the boundary, respectively. The temperature can then be obtained by using (18), which is recast in a general format as:

$$\frac{T(x, t)}{T_m} = \frac{T_0(x, t)}{T_m} \pm \frac{\hat{H}(t - t_0)}{Ste} \int_0^{t - t_0} \frac{\mathrm{d}R(\tau + t_0)}{\mathrm{d}\tau} G(x, t \,|\, R(\tau + t_0), \tau)\, \mathrm{d}\tau \tag{21}$$

where all temperatures, including T_m, are measured in excess of the initial temperature, and

$$T_0(x, t) = \sqrt{\frac{\alpha}{\pi}} \int_0^t \frac{E(s)}{(t - s)^{1/2}} \exp\left[-\frac{x^2}{4\alpha(t - s)} \right] \mathrm{d}s \tag{22}$$

in which

$$E(s) = \begin{cases} \dfrac{x}{2\alpha} \dfrac{F(s)}{t-s} \\ \dfrac{1}{k} G(s) \end{cases} \qquad (23a, b)$$

Here $F(s)$ and $G(s)$ denote respectively the temperature and heat flux condition sought. Also

$$\hat{H}(t-t_0) = \begin{cases} 1 & \text{for} \quad t > t_0 \\ 0 & \quad t \leqslant t_0 \end{cases} \qquad (24)$$

$$Ste = \frac{cT_m}{L} \qquad (25)$$

where Ste is known as the Stefan number. With the use of the circumflexed Heaviside function given by (24), (21) holds for all time and for both one- and two-phase problems.

Equation (21) can be used to determine the unknown boundary condition by invoking use of the condition at the interface as:

$$\pm\left[1 - \frac{T_0(R(t), t)}{T_m}\right] = \frac{\hat{H}(t-t_0)}{Ste} \int_0^{t-t_0} \frac{\mathrm{d}R(\tau + t_0)}{\mathrm{d}\tau} G(R(t), t - t_0 | R(\tau + t_0), \tau)\, \mathrm{d}\tau \qquad (26)$$

where the plus and minus signs on the left hand side are to be used for freezing and melting, respectively.

NUMERICAL SOLUTION

A local linearization can be used to solve (26) numerically. In this effort, the entire time range is divided into small increments in which the interface position is taken to be linear (see *Figure 2*). Then $\mathrm{d}R/\mathrm{d}t$ can be taken out of the integral for each increment, and the convolution integral written as a summation as:

$$\pm \frac{T_0(R(t_N), t_N)}{T_m} = \pm 1 - \frac{\hat{H}(t_N - t_0)}{Ste} \sum_{n=1}^{N} \left[\frac{\mathrm{d}R(t_n^-)}{\mathrm{d}t} \int_{t_{n-1}-t_0}^{t_n - t_0} G(R(t_N), t_N - t_o | R(\tau + t_0), \tau)\, \mathrm{d}\tau\right] \qquad (27)$$

Here the signs are to be selected following the statement below (26). For the inverse Stefan problems, the right-hand side can be evaluated with the input of the interface motion data. As for the left-hand side, if the boundary conditions are represented by a power series, then the number of terms on this side must be equal to the number of the terms that are taken in the series. In practice, (27) can be written by means of matrix elements for a melting problem for the series method as:

$$\sum_{n=1}^{N} b_{mn} a_n = c_m + \sum_{n=1}^{N} d_{mn} e_n \qquad (28)$$

where $m = N$; $t_N > t_0$; $N = 1, 2, \ldots, N$; and

$$b_{mn} = \begin{cases} 0 & \text{for } m < n \\ \dfrac{R(t_m)}{2\alpha} \displaystyle\int_0^{t_m} \frac{s^{n-1}}{(t_m - s)^{3/2}} \exp\left[-\frac{R(t_m)^2}{4\alpha(t_m - s)}\right] \mathrm{d}s & \text{for } m \geqslant n; F(s) = \displaystyle\sum_{n=1}^{N} a_n s^{n-1} \\ \dfrac{1}{k} \displaystyle\int_0^{t_m} \frac{s^{n-1}}{(t_m - s)^{1/2}} \exp\left[-\frac{R(t_m)^2}{4\alpha(t_m - s)}\right] \mathrm{d}s & \text{for } m \geqslant n; G(s) = \displaystyle\sum_{n=1}^{N} a_n s^{n-1} \end{cases} \qquad (29a,b,c)$$

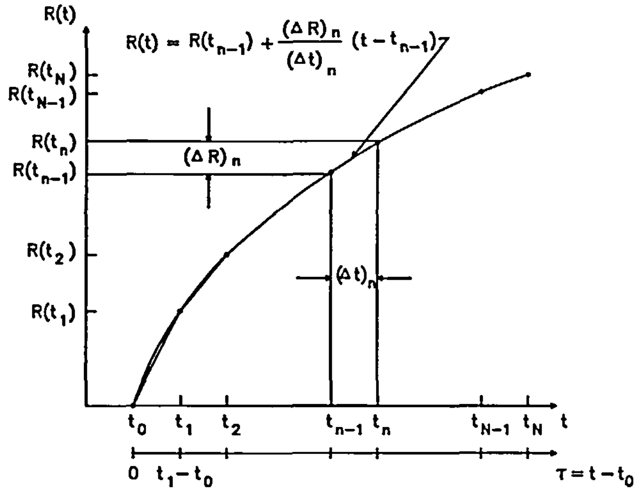

Figure 2 Linearization of the interface position curve

$$c_m = \sqrt{\frac{\pi}{\alpha}}\,T_m \tag{30}$$

$$d_{mn} = \begin{cases} 0 & \text{for } m < n \\ \dfrac{L}{c}\dfrac{\pi}{\alpha}\displaystyle\int_{t_{n-1}-t_0}^{t_n-t_0} G(R(t_m), t_m - t_0 \,|\, R(\tau + t_0), \tau)\,\mathrm{d}\tau & \text{for } m \geqslant n \end{cases} \tag{31a,b}$$

$$e_n = \frac{\mathrm{d}R(t_n^-)}{\mathrm{d}t} \tag{32}$$

Notice that the integrals in these equations can be performed in closed form if the missing conditions are expressed in the power series as shown in (29) or in terms of a Fourier series. The coefficient matrices B and D are of a lower triangular structure, a characteristic permitting the solution of (28) to be carried out simply by using forward substitution, a simple numerical procedure.

The boundary conditions can also be determined by use of an incremental approach. In this method, (27) is recast as:

$$\pm\int_{t_{N-1}}^{t_N} \frac{E(s)}{(t_N - s)^{1/2}} \exp\left[-\frac{R(t_N)^2}{4\alpha(t_N - s)}\right] \mathrm{d}s$$
$$= \sqrt{\frac{\pi}{\alpha}}\,T_m\left\{\pm 1 - \frac{\hat{H}(t_N - t_0)}{Ste}\sum_{n=1}^{N}\left[\frac{\mathrm{d}R(t_n^-)}{dt}\int_{t_{n-1}-t_0}^{t_n-t_0} G(R(t_N), t_N - t_0 \,|\, R(\tau + t_0), \tau)\,\mathrm{d}\tau\right]\right\}$$
$$-\left\{\pm\sum_{n=1}^{N-1} \frac{E(s)}{(t_N - s)^{1/2}} \exp\left[-\frac{R(t_N)^2}{4\alpha(t_N - s)}\right] \mathrm{d}s\right\} \tag{33}$$

where t_0 is set to zero all the time for the integral on the left and the last integral on the right. In practice, (33) can be recast as:

$$\frac{F(t_N)}{G(t_N)} = \begin{pmatrix} P_{N,F} \\ P_{N,G} \end{pmatrix}^{-1} \left\{\sqrt{\frac{\pi}{\alpha}}\,T_m + \sum_{n=1}^{N} d_{Nn}e_n - \sum_{n=1}^{N-1} \frac{F(t_n)P_{n,F}}{G(t_n)P_{n,G}}\right\} \tag{34a,b}$$

where $t_N > t_0$; $N = 1, 2, \ldots, N$; d_{Nn} can be obtained by referring to (31); and

$$\begin{matrix} P_{n,F} \\ P_{n,G} \end{matrix} = \begin{cases} \dfrac{R(t_N)}{2\alpha} \displaystyle\int_{t_{n-1}}^{t_n} \frac{1}{(t_N - s)^{3/2}} \exp\left[-\frac{R(t_N)^2}{4\alpha(t_N - s)} \right] ds \\ \dfrac{1}{k} \displaystyle\int_{t_{n-1}}^{t_n} \frac{1}{(t_N - s)^{1/2}} \exp\left[-\frac{R(t_N)^2}{4\alpha(t_N - s)} \right] ds \end{cases} \quad (35a,b)$$

In (34), the summation vanishes when $N - 1 = 0$. Then starting from $N = 1, 2$, and so on, the conditions can be evaluated incrementally. In this effort, the F and G values found from the previous time steps are used as input in the computation of the right-hand side of (34), which immediately gives the F and G values for the succeeding new time step. This continues until the desired time is reached. The algorithms can be developed for a rapid solution of the conditions.

CRITIQUE OF THE METHOD

Using Green's functions, the present method is limited to the solution of problems whose Green's function can be obtained analytically. This excludes those problems whose boundary conditions and governing equations are non-linear and such non-linearity cannot be resolved by using transformation (e.g. Kirchhoff transformation). Yet with the use of Green's function, it projects an impression of being related to the boundary element method that has been undergoing thorough development in recent years[17]. Yet, they are functionally different. In the boundary element method, the boundary element equations are written separately for the liquid and solid regions, whereas in the present method, only one equation (27) is derived. The number of the equations to be solved in the present method is thus reduced by half, which is particularly true in the solution by the incremental approach described earlier.

More important, in the boundary element method, the problems cannot be solved without the information for the heat fluxes that appear on both sides of the interface[17,18]. Such fluxes, however, are unnecessary for solution in the present work. Instead, (26) embodies conditions (12) and (13) in the form of a single integrodifferential equation. There is no need for the satisfaction of the flux conditions; as a result, the present method is more effective because it requires less information for input.

As will be shown in the next section, the present method is also accurate. In fact, such accuracy is not unexpected because (26) is exact. The only approximation in the analysis is the local linearization which has been applied to the interface motion and the boundary condition; see (33) and (35). In fact, in the present analysis, they have been approximated by using constant elements. The analysis can be improved for accuracy by using higher order elements, such as linear and quadratic elements as in References 3 and 19.

RESULTS AND DISCUSSION

For the numerical experiments performed in this study, the interface motion data are taken from Reference 20. Aluminium will be used for tests; its melting point is 932 K. The inverse solution techniques developed in this paper are used to solve six example problems of which four having exact solutions. The retrieved conditions for these examples can thus be compared with the exact solutions for error. In these four examples, the interfaces move as a function of square root of time, and the interface motion data are used to retrieve the constant temperature conditions that appear on the surface. This problem is known as the Stefan–Neumann problem, and *Table 2* provides a summary of the conditions tested in all the examples.

The first example deals with a general two-phase Stefan–Neumann problem; the medium is

Table 2 Conditions tested in six examples

Problem description		Input data	True condition $F(t)$ (K); $G(t)$ (W/m^2)	Type of boundary condition: Assumed	Solved
1	two-phase $T_i = 300\text{ K} < T_m$	$R(t) \sim \sqrt{t}$	$F(t) = 1000$	temperature	temperature
2*	two-phase $T_i = 300\text{ K} < T_m$	$R(t) \sim \sqrt{t}$	$F(t) = 1000$	heat flux	temperature
3	one-phase $T_i = 932\text{ K} = T_m$	$R(t) \sim \sqrt{t}$	$F(t) = 1000$	temperature	temperature
4*	one-phase $T_i = 932\text{ K} = T_m$	$R(t) \sim \sqrt{t}$	$F(t) = 1000$	heat flux	temperature
5	one-phase $T_i = 932\text{ K} = T_m$	Reference 20	$F(t) = 1000 + 5t$	temperature	temperature
6	two-phase $T_i = 300\text{ K} < T_m$	Reference 20	$G(t) = 6.39 \times 10^6 + 2.8229 \times 10^5 t$	heat flux	heat flux

*This example serves to show that the results obtained are independent of the condition initially assumed

initially subcooled to 300 K, which is lower than the melting temperature (see *Table 2*). For this example, the temperature at the boundary is unknown; an unknown temperature is thus assumed to appear at the boundary, and the interface motion data are used to find this temperature. The results are listed in *Table 3*, where two sets of results are given—one for the series solution method and the other for the incremental solution method.

In the series solution method, a power series of degree $N-1$ is used to represent the temperature (see (29b)), and two values of N are tested. In this method, the interface position data at equal time intervals are used for input. Thus, for example, for $N = 3$, R data at times equal to 0, 6, 12, and 18 sec are used to determine the series, which is, in turn, used to generate the temperature values listed for 20 time steps in the Table. Comparing the retrieved temperatures at the boundary by both of the series with the true condition (exact temperature) listed to the left indicates they are in good agreement. In this case, the coefficients found for these series are listed at the bottom of the Table. From the temperature values tabulated, the results for the low power series appear to be as good as those of the high power series, and such trend persists even with the test of a higher power series of degree 10 (results not shown). This gives the indication that the temperatures have been converged. As for the incremental solution method, the results are also good; errors are of the order of $10^{-3}\%$ at large time. For the incremental method, the boundary conditions are found at exactly the same times when the interface positions are given. The time step for the solution is thus identical to that for the interface data input. Also notice that the convergence and stability that are normally encountered in the conventional finite difference methods are non-existent in the present incremental solution of integral equations. Also as in the case of the series solution method, the incremental solution results are generally better at large time than small time, which will be further discussed later.

In the example above, a temperature condition is imposed, and the same 'type' of condition is assumed in the process of the inverse solution. Since the type of the condition that is imposed on the boundary is unknown *a priori*, it may well be the heat flux condition that one assumes, and the question to be addressed now is whether it is still possible to retrieve the temperature condition via the assumed flux condition. Use will now be made of (21) to determine this temperature condition once the boundary flux condition is found, and the results are listed in

Table 3 Comparison between true and retrieved conditions for first Stefan–Neumann example problem solved for boundary temperature

Time (sec)	Imposed true condition	Boundary conditions, T (K) — Retrieved — Series method $N = 3$	Series method $N = 4$	Incremental method
1	1000	1003.22	1003.17	1003.49
2	1000	1002.97	1002.87	1000.41
3	1000	1002.72	1002.58	1000.20
4	1000	1002.48	1002.30	1000.11
5	1000	1002.25	1002.04	1000.03
6	1000	1002.03	1001.79	1000.05
7	1000	1001.82	1001.56	1000.04
8	1000	1001.61	1001.34	1000.02
9	1000	1001.42	1001.14	1000.68
10	1000	1001.23	1000.95	999.88
11	1000	1001.06	1000.78	999.97
12	1000	1000.89	1000.63	999.99
13	1000	1000.73	1000.50	999.99
14	1000	1000.58	1000.38	999.99
15	1000	1000.43	1000.28	999.99
16	1000	1000.30	1000.20	999.98
17	1000	1000.17	1000.14	999.98
18	1000	1000.06	1000.09	999.98
19	1000	999.95	1000.07	999.90
20	1000	999.85	1000.07	999.94

$F(s) = \sum_{n=1}^{N} a_n s^{n-1}$

For $N = 3$, $a_1 = 1003.490405$
$a_2 = -0.268732$
$a_3 = 4.350448 \times 10^{-3}$

For $N = 4$, $a_1 = 1003.490405$
$a_2 = -0.322478$
$a_3 = 6.264645 \times 10^{-3}$
$a_4 = 6.552981 \times 10^{-5}$

Table 4. As shown in the Table, the incremental solution results are still good but the series solution results are not as accurate as those listed in *Table 3*. This is certainly a result of the errors being accumulated first in the evaluation of the heat flux next in the evaluation of the temperature using the previously determined heat flux. While this example serves well to illustrate that the conditions are still exchangeable, such a two-step solution of the condition may lead to large errors, particularly in the series solution method and should thus be avoided in practice. In fact, according to experience, the computer time saved in this two-step approach of solving flux then temperature is insignificant as compared with that in the separate, one-step approach of direct evaluation of the flux and temperature.

A slight modification is made in the next two examples: this time the medium is not subcooled, the initial temperature being equal to the melting temperature of the medium (see examples 3 and 4 and *Table 2*). The Stefan–Neumann problems solved thus become a one-phase problem. It will be tested that the results are unaffected by the change to the one-phase problem. Again, the same series of tests are made and the results are listed in *Tables 5* and *6*. Again, the one-step solution results are good (*Table 5*). The two-step solution results are poorer (*Table 6*), further reinforcing the recommendation made earlier in the testing of the two-phase problems.

Table 4 Comparison between true and retrieved conditions for second Stefan–Neumann example problem solved for boundary flux then temperature

	Boundary conditions, T (K)			
		Retrieved		
		Series method		
Time (sec)	Imposed true condition	$N = 4$	$N = 5$	Incremental method
1	1000	585.89	620.85	1023.06
2	1000	715.58	758.46	1007.71
3	1000	799.99	844.90	1003.60
4	1000	862.06	905.50	1000.88
5	1000	909.11	948.70	1000.04
6	1000	945.07	979.11	999.68
7	1000	972.34	999.67	999.52
8	1000	992.57	1012.44	999.44
9	1000	1006.96	1019.03	998.11
10	1000	1016.46	1020.76	998.30
11	1000	1021.85	1018.76	998.37
12	1000	1023.76	1014.02	998.44
13	1000	1022.76	1007.48	998.51
14	1000	1019.34	999.99	998.59
15	1000	1013.95	992.40	998.66
16	1000	1007.01	985.51	999.45
17	1000	998.89	980.13	999.16
18	1000	989.95	977.04	999.11
19	1000	980.54	977.04	999.07
20	1000	971.00	980.93	997.15

$G(s) = \sum_{n=1}^{N} a_n s^{n-1}$

For $N = 4$, $a_1 = 8001187.956002$
$a_2 = -387201.858263$
$a_3 = 1581.702473$
$a_4 = 148.390536$

For $N = 5$, $a_1 = 8945600.085189$
$a_2 = -541131.047558$
$a_3 = 2763.120508$
$a_4 = 324.034498$
$a_5 = 3.817008$

Having satisfactorily completed testing of the Stefan–Neumann problems, attention is now directed to the solution of inverse Stefan problems whose interface motions must be met by the imposing time-variant temperature and flux conditions. There are no exact solutions for these problems, and the interface motion data are taken from Choi[20] who have solved the regular (forward) version of the problems with great accuracy. The interface position data are then used to retrieve the boundary conditions and the results are listed in *Tables 7* and *8*. *Table 7* gives the results for the direct retrieval of the linear temperature condition:

$$F(t) = 1000 + 5t \tag{36}$$

while *Table 8* gives the results for the direct retrieval of the linear flux condition:

$$G(t) = 6.39 \times 10^6 + 2.8229 \times 10^5 \tag{37}$$

As described in *Table 2*, those in *Table 7* are for the medium initially at the phase change temperature, while those in *Table 8* are for the medium initially subcooled at 300 K. The former is thus a one-phase problem while the latter is a two-phase problem. Again both series and

Table 5 Comparison between true and retrieved conditions for third Stefan–Neumann example problem solved for boundary temperature

	Boundary conditions, T (K)			
		Retrieved		
		Series method		
Time (sec)	Imposed true condition	$N = 3$	$N = 4$	Incremental method
1	1000	1033.83	1033.12	1037.49
2	1000	1030.31	1028.96	998.22
3	1000	1026.93	1025.01	998.58
4	1000	1023.69	1021.29	998.85
5	1000	1020.59	1017.80	999.05
6	1000	1017.63	1014.55	999.20
7	1000	1014.81	1011.55	999.24
8	1000	1012.13	1008.81	999.38
9	1000	1009.59	1006.33	999.44
10	1000	1007.19	1004.13	999.49
11	1000	1004.93	1002.21	999.53
12	1000	1002.81	1000.57	999.59
13	1000	1000.82	999.23	999.60
14	1000	998.98	998.20	999.65
15	1000	997.28	997.47	999.65
16	1000	995.71	997.07	999.69
17	1000	994.29	997.00	999.69
18	1000	993.00	997.26	999.73
19	1000	991.86	997.87	999.71
20	1000	990.85	998.82	999.78

$F(s) = \sum_{n=1}^{N} a_n s^{n-1}$

For $N = 3$, $a_1 = 1037.496339$
$a_2 = -3.728416$
$a_3 = 6.982696 \times 10^{-2}$

For $N = 4$, $a_1 = 1037.496324$
$a_2 = -4.474102$
$a_3 = 1.005513 \times 10^{-1}$
$a_4 = 1.324365 \times 10^{-3}$

incremental solution methods are used for solution and their results are good. For example in *Table 7*, the series solution results converge even with a value of N that is as low as 3, whereas in seeking the flux condition in *Table 8*, the series converges rapidly from $N = 4$ to $N = 5$ (higher degree results not shown). In *Table 7* the medium melts as soon as the boundary condition is imposed, whereas in *Table 8* the medium starts to melt at time greater than 4 sec. In either case, the accuracy of the results appears to be unaffected by the time when melting takes place. Other examples are given in Reference 25. Tests for the time variant conditions are thus successful.

The inverse solution techniques developed in this paper are expected to be accurate as mentioned earlier that is close to the end of the previous section. Yet for the Stefan problems solved in this paper, the accuracy of the techniques is better at large time than small time. This can be attributed to the curvature of the interface position curve, which is usually large at small time[15] (see *Figure 2*). Then, in the numerical solution of (27), there will be a slight error associated with the linearization of the position at small time. At large time, however, the position curve tends to be linear; the linearization error will be diminished. In fact, as time progresses, the accurate terms under the summation in (27) rapidly outnumber the inaccurate terms to the effect that the boundary conditions can always be evaluated accurately with the present method at large time; see the results in *Tables 3* through *8*. This is a distinct departure from the trends

Table 6 Comparison between true and retrieved conditions for fourth Stefan–Neumann example problem solved for boundary flux then temperature

	Boundary conditions, T (K)			
		Retrieved		
		Series method		
Time (sec)	Imposed true condition	$N=4$	$N=5$	Incremental method
1	1000	906.32	917.43	1056.39
2	1000	953.87	966.25	996.60
3	1000	980.48	991.90	1002.26
4	1000	997.67	1006.84	999.33
5	1000	1008.95	1015.10	999.70
6	1000	1016.02	1018.67	999.45
7	1000	1019.89	1018.86	999.16
8	1000	1021.28	1016.61	999.22
9	1000	1020.73	1012.66	999.09
10	1000	1018.66	1007.67	999.23
11	1000	1015.44	1002.18	999.05
12	1000	1011.39	996.73	999.32
13	1000	1006.79	991.80	998.89
14	1000	1001.89	987.87	999.72
15	1000	996.94	985.38	998.27
16	1000	992.16	984.80	1001.19
17	1000	987.77	986.56	994.04
18	1000	983.98	991.12	1014.19
19	1000	980.99	998.92	951.97
20	1000	979.00	1010.41	1162.46

$G(s) = \sum_{n=1}^{N} a_n s^{n-1}$

For $N = 4$, $a_1 = 2805963.114624$
$a_2 = -204888.890714$
$a_3 = 2268.882336$
$a_4 = 99.892255$
For $N = 5$, $a_1 = 3137161.977178$
$a_2 = -286340.839757$
$a_3 = 3963.574559$
$a_4 = 218.130426$
$a_5 = 1.467676$

of other time marching schemes reported in the literature in which the errors tend to grow with time. It should also be pointed out that, for the Stefan–Neumann problem chosen for comparison in the present study, there is a singularity of the temperature at zero time. This also contributes to the large discrepancy of the results at small time, which must not be overlooked.

It has been firmly established that, in the solution of the Stefan problems, only the Stefan–Neumann problems can be solved exactly. Yet, it is also possible to develop an exact solution for an exponential condition imposed on the boundary; such condition, however, has been considered as physically untenable in the literature[21]. Worse yet, such a condition gives rise to a constant velocity of the interface, a situation making the present linearization scheme exact in the solution of the inverse problems[20]. Perfect results will be obtained, rendering the test of the exponential conditions meaningless.

EXTENSION AND CONCLUDING REMARKS

The analysis developed in this paper can be readily extended for the solution of inverse problems with multiple phases. For such problems, times for re-melt and re-freeze of the medium must

Table 7 Comparison between true and retrieved conditions for fifth example problem solved for boundary temperature

Time (sec)	Imposed true condition	Boundary condition, T (K) — Retrieved — Series method $N=3$	Error (%)	$N=4$	Error (%)	Incremental method	Error (%)
1	1005	1073.05	6.77	1078.13	7.28	1003.61	0.13
2	1010	1064.27	5.37	1061.16	5.07	1007.83	0.21
3	1015	1056.80	4.11	1048.37	3.29	1012.50	0.24
4	1020	1050.64	3.00	1039.36	1.90	1017.33	0.26
5	1025	1045.81	2.03	1033.71	0.85	1022.27	0.26
6	1030	1042.29	1.19	1031.04	0.10	1027.17	0.27
7	1035	1040.09	0.49	1030.94	0.39	1032.24	0.26
8	1040	1039.21	0.07	1033.01	0.67	1037.26	0.26
9	1045	1039.64	0.51	1036.86	0.78	1042.29	0.25
10	1050	1041.40	0.81	1042.07	0.75	1047.31	0.25
11	1055	1044.47	0.99	1048.26	0.64	1052.35	0.25
12	1060	1048.86	1.05	1055.02	0.47	1057.36	0.24
13	1065	1054.56	0.97	1061.95	0.29	1062.38	0.24
14	1070	1061.59	0.78	1068.65	0.13	1067.38	0.24
15	1075	1069.93	0.47	1074.72	0.02	1072.39	0.24
16	1080	1079.58	0.03	1079.77	0.02	1077.40	0.23
17	1085	1090.56	0.51	1083.38	0.15	1082.37	0.24
18	1090	1102.85	1.17	1085.17	0.44	1087.50	0.22
19	1095	1116.47	1.96	1084.72	0.94	1092.06	0.26
20	1100	1131.39	2.85	1081.65	1.67	1098.80	0.10

$F(s) = \sum_{n=1}^{N} a_n s^{n-1}$

For $N = 3$, $a_1 = 1083.1636095$
$a_2 = -10.764038059$
$a_3 = 0.65879192310$
For $N = 4$, $a_1 = 1099.6680063$
$a_2 = -23.956349650$
$a_3 = 2.4866623756$
$a_4 = -0.0666941076$

be closely accounted for, and the analyses described in References 16 and 22 can be readily adapted for the development of the inverse solution techniques presented in this paper. The present analysis can also be extended for the solution of inverse problems in which the properties are unequal for different phases of the medium. For such problems, double source and sink fronts must be used as given in the solution of the regular problems in Reference 23. Finally, it is noted that although problems in one-dimensional, semi-infinite domain have been solved for examples in this paper, problems in finite domains (e.g. plane walls) can also be solved with the present methods. For these problems, there are two boundaries and two boundary conditions are imposed. Two unknowns are thus sought simultaneously, and this requires the input of one additional condition in the form of either temperature or heat flux at any interior point close to the boundary where no phase change takes place. On the other hand, for problems with two phase-change interfaces caused by separate heat input simultaneously from both sides of the boundaries, the interface motion data for the second interface will serve as this additional condition. In any event, no flux information is needed at both sides of the interfaces. Furthermore, the present methods can also be applied to the solution of problems in multiple dimensions. Again the inverse solution for these problems can be developed on the basis of the solution of the regular (forward) versions of these problems. Solutions of the regular, two-dimensional Stefan problems by the source-and-sink method have been given in Reference 26.

Table 8 Comparison between true and retrieved conditions for sixth example problem solved for boundary heat flux

		Boundary condition, q (W/m^2)					
		Retrieved					
	Imposed	Series method					
Time (sec)	true condition	$N = 4$	Error (%)	$N = 5$	Error (%)	Incremental method	Error (%)
4.25	7589730.94	8076200.72	6.41	7558675.50	−0.40	7184872.69	5.33
4.50	7660418.94	8147001.48	6.35	7694176.58	0.44	7371675.43	3.77
4.75	7731106.94	8206716.05	6.15	7815212.59	1.08	7493717.81	3.07
5.00	7801794.94	8257180.74	5.83	7920261.67	1.52	7596454.24	2.63
5.25	7872482.94	8300231.83	5.43	8008672.53	1.73	7689904.72	2.31
5.50	7943170.94	8337705.60	4.96	8080664.49	1.73	7775533.53	2.11
5.75	8013858.94	8371438.35	4.46	8137327.48	1.54	7859641.15	1.92
6.00	8084546.94	8403266.35	3.94	8180622.00	1.19	7940899.93	1.77
6.25	8155234.94	8435025.90	3.43	8213379.15	0.71	8020420.49	1.65
6.50	8225922.94	8468553.29	2.94	8239300.64	0.16	8098888.64	1.54
6.75	8296610.94	8505684.79	2.52	8262958.75	0.40	8176244.61	1.45
7.00	8367298.94	8548256.71	2.16	8289796.37	0.93	8252784.49	1.36
7.25	8437986.94	8598105.32	1.89	8326126.99	1.33	8329871.72	1.28
7.50	8508674.94	8657066.91	1.74	8379134.69	0.61	8402584.80	1.25
7.75	8579362.94	8726977.77	1.72	8456874.13	1.42	8480061.68	1.15
8.00	8650050.94	8809674.19	1.84	8568270.59	0.94	8553180.08	1.12
8.25	8720738.94	8906992.45	2.13	8723119.93	0.02	8628203.50	1.06
8.50	8791426.94	9020768.84	2.60	8932088.60	1.60	8701314.04	1.02
8.75	8862114.94	9152839.65	3.28	9206713.65	3.88	8775467.98	0.98
9.00	8932802.94	9305041.17	4.16	9559402.73	7.01	8847997.32	0.95

$G(s) = \sum_{n=1}^{N} a_n s^{n-1}$

For $N = 4$, $a_1 = 3397040.84$
$a_2 = 2247925.15$
$a_3 = -353114.747$
$a_4 = 19587.0623$
For $N = 5$, $a_1 = 8734725.30$
$a_2 = -3092029.40$
$a_3 = 1293821.95$
$a_4 = -188030.644$
$a_5 = 9286.34410$

REFERENCES

1 Hsieh, C. K. and Lin, J. Solution of inverse heat-conduction problems with unknown initial conditions, *Proc. Eighth Int. Heat Transfer Conf.*, **2**, 609–614 (1986)

2 Beck, J. V., Blackwell, B. and St. Clair, C. R. Jr. *Inverse Heat Conduction, Ill-posed Problems*, Wiley-Interscience, New York (1985)

3 Hsieh, C. K., Choi, C.-Y. and Liu, K. M. A domain extension method for quantitative detection of cavities by infrared scanning, *J. Nondestr. Eval.*, **8** (3), 195–211 (1989)

4 Chen, M. M., Pederson, C. O. and Chato, J. C. On the feasibility of obtaining three-dimensional information from thermographic measurements, *J. Biomech. Eng.*, **99**, 58–64 (1978)

5 Nikitenko, N. I. and Kolodnyi, Y. M. Numerical solution of the inverse heat-conduction problem for determining thermal constants, *Inzh.-Fiz. Zh.*, **33**, 1058–1061 (1977)

6 Frederick, D. and Grief, R. A method for the solution of heat transfer problems with a change of phase, *J. Heat Transfer*, **107**, 520–526 (1985)

7 Peel, D. A. Some moving boundary problems in the steel industry, in *Moving Boundary Problems in Heat Flow and Diffusion*, (Eds. J. R. Ockendon and W. R. Hodgkins), Clarendon Press, Oxford (1975)

8 Landram, C. S. Measurement of fusion boundary energy transport during arc welding, *J. Heat Transfer*, **105**, 550–554 (1983)

9 Abuaf, N. and Gutfinger, C. Heat transfer with a moving boundary, application to fluidized bed coating of thin plates, *Int. J. Heat Mass Transfer*, **16**, 213–216 (1973)

10 Rubinsky, B. and Shitzer, A. Analysis of a Stefan-like problem in a biological tissue around a cryosurgical probe, *J. Heat Transfer*, **98**, 514–519 (1976)
11 Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics*, McGraw-Hill, New York (1953)
12 Ozisik, N. M. *Heat Conduction*, Wiley, New York (1980)
13 Hsieh, C. K. and Shang, H. Solution of boundary value heat conduction problems with variable convective coefficients by a boundary condition dissection method, *Nucl. Eng. Design*, **110**, 17–31 (1988)
14 Lightfoot, N. M. H. Solidification of molten steel, *Proc. Lond. Math. Soc.*, (*Ser. 2*), **31**, 97–116 (1929)
15 Hsieh, C. K. and Choi, C.-Y. Solution of one- and two-phase melting and solidification problems imposed with constant or time-variant temperature and flux boundary conditions, *J. Heat Transfer*, **114**, 524–528 (1992)
16 Choi, C.-Y. and Hsieh, C. K. Solution of Stefan problems imposed with cyclic temperature and flux boundary conditions, *Int. J. Heat Mass Transfer* **35**, 1181–1195 (1992)
17 Rubinsky, B. and Shitzer, A. Analytical solutions to the heat equation involving a moving boundary with applications to the change of phase problem (the inverse Stefan problem), *J. Heat Transfer*, **100**, 300–304 (1978)
18 Zabaras, N., Mukherjee, S. and Richmond, O. An analysis of inverse heat transfer problems with phase changes using an integral method, *J. Heat Transfer*, **110**, 554–561 (1988)
19 Brebbia, C. A. *The Boundary Element Techniques in Engineering*, Newnes-Butterworth, Boston (1980)
20 Choi, C.-Y. Exact and numerical solution of one- and two-phase melting and solidification problems imposed with constant or time-variant temperature and flux conditions, *PhD Dissertation*, University of Florida (1991)
21 Carslaw, H. S. and Jaeger, J. C. *Conduction of Heat in Solids*, Clarendon Press, Oxford (1959)
22 Hsieh, C. K. and Choi, C.-Y. A general analysis of phase change energy storage for solar energy applications, *J. Solar Energy Eng.* (in press)
23 Kolodner, I. I. Free boundary problem for the heat equation with applications to problems of change phase, *Commun. Pure Appl. Math.*, **9**, 1-31 (1956)
24 Patel, P. D. Interface conditions in heat conduction problems with change of phase, *AIAA J.*, **6**, 2454 (1968)
25 Akbari, M. Source-and-sink method of solution of moving boundary problems, *PhD Dissertation*, University of Florida (1992)
26 Li, H. Source-and-sink method of solution of two- and three-dimensional Stefan problems, *PhD Dissertation*, University of Florida (1992)

APPENDIX

For a Stefan problem in two-dimensional Cartesian system, the interface position can be represented as:

$$y_f = R(x_f, t) \tag{36}$$

Then, according to References 12 and 24:

$$v_n(t)\delta(\bar{r} - \bar{r}_f) = -\frac{\partial R}{\partial t}\delta(y - y_f) \tag{37}$$

Three-dimensional cases and problems in other coordinate systems can be formulated accordingly.